BASIC POLYNOMIAL INVARIANTS, FUNDAMENTAL REPRESENTATIONS AND THE CHERN CLASS MAP

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Introduction

Consider a crystallographic root system Φ together with its Weyl group W acting on the weight lattice Λ of Φ. Let $\mathbb{Z}[\Lambda]^W$ and $S^*(\Lambda)^W$ be the W-invariant subrings of the integral group ring $\mathbb{Z}[\Lambda]$ and the symmetric algebra $S^*(\Lambda)$. A celebrated theorem of Chevalley says that $\mathbb{Z}[\Lambda]^W$ is a polynomial ring over $\mathbb{Z}$ in classes of fundamental representations $\rho_1, \ldots, \rho_n$ and $S^*(\Lambda)^W \otimes \mathbb{Q}$ is a polynomial ring over $\mathbb{Q}$ in basic polynomial invariants $q_1, \ldots, q_n$, where $n = \mathrm{rank}(\Phi)$.

In the present paper we establish and investigate the relationship between ρ_i's and q_i's. To do this we introduce an equivariant analogue of the Chern class map ϕ_i that provides an isomorphism between the truncated rings $\mathbb{Z}[\Lambda]/I_m^j$ and $S^*(\Lambda)/I_a^j$ modulo powers of the respective augmentation ideals. This allows us to express basic polynomial invariants in terms of fundamental representations and vice versa, hence relating the geometry of the variety of Borel subgroups X with representation theory of the respective Lie algebra $\mathfrak{g}$.

A multiple of ϕ_i restricted to the respective cohomology (K_0 and CH^*) of X gives the classical Chern class map $c_i\colon K_0(X) \to CH^i(X)$. This geomeric interpretation provides a powerful tool to compute the annihilators of the torsion of the Grothendieck γ-filtration on K_0 of twisted forms of X as well as a tool to estimate the torsion part of its Chow groups in small codimensions.

The paper is organized as follows. In the first section we introduce the I-adic filtrations on $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$ together with an isomorphism ϕ_i on their truncations. Then we study the subrings of invariants and introduce the key notion of an exponent τ_i of a W-action on a free abelian group Λ. Roughly speaking, the integers τ_i measure how far is the ring $S^*(\Lambda)^W$ (with integer coefficients) from being a polynomial ring in q_i's. In section 5 we compute all the exponents up to degree 4 and show that they all coincide with the Dynkin index of the Lie algebra $\mathfrak{g}$. In section 6 we apply the obtained results to estimate the torsion in Grothendieck γ-filtration of some twisted flag varieties. In section 7 we compute the second exponent τ_2 for a non-crystallographic group H_2.

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1. Two filtrations

Consider the two covariant functors $S^*(-)$ and $\mathbb{Z}[-]$ from the category of abelian groups to the category of commutative rings

$$S^*(-)\colon \Lambda \mapsto S^*(\Lambda) \text{ and } \mathbb{Z}[-]\colon \Lambda \mapsto \mathbb{Z}[\Lambda]$$

given by taking the symmetric algebra of an abelian group Λ and the integral group ring of Λ respectively. The ith graded component $S^i(\Lambda)$ is additively generated by monomials $\lambda_1\lambda_2\ldots\lambda_i$ with $\lambda_j \in \Lambda$ and the ring $\mathbb{Z}[\Lambda]$ is additively generated by exponents e^λ, $\lambda \in \Lambda$.

The trivial group homomorphism induces the ring homomorphisms

$$\epsilon_a\colon S^*(\Lambda) \to \mathbb{Z} \text{ and } \epsilon_m\colon \mathbb{Z}[\Lambda] \to \mathbb{Z}$$

called the augmentation maps. By definition ϵ_a sends every element of positive degree to 0 and ϵ_m sends every e^λ to 1. Let I_a and I_m denote the kernels of ϵ_a and ϵ_m respectively. Observe that $I_a = S^{>0}(\Lambda)$ consists of elements of positive degree and I_m is generated by differences $(1-e^{-\lambda})$, $\lambda \in \Lambda$. Consider the respective I-adic filtrations:

$$S^*(\Lambda) = I_a^0 \supseteq I_a \supseteq I_a^2 \supseteq \ldots \text{ and } \mathbb{Z}[\Lambda] = I_m^0 \supseteq I_m \supseteq I_m^2 \supseteq \ldots$$

and let

$$gr_a^*(\Lambda) = \bigoplus_{i\geq 0} I_a^i/I_a^{i+1} \text{ and } gr_m^*(\Lambda) = \bigoplus_{i\geq 0} I_m^i/I_m^{i+1}$$

denote the associated graded rings. Observe that $gr_a^*(\Lambda) = S^*(\Lambda)$.

1.1. **Example.** If $\Lambda \simeq \mathbb{Z}$, then the ring $S^*(\Lambda)$ can be identified with the polynomial ring in one variable $\mathbb{Z}[\omega]$, where ω is a generator of Λ and the ring $\mathbb{Z}[\Lambda]$ can be identified with the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$ where $x = e^\omega$. The augmentations ϵ_a and ϵ_m are given by

$$\epsilon_a\colon \omega \mapsto 0 \text{ and } \epsilon_m\colon x \mapsto 1.$$

We have $I_a = (\omega)$ and I_m is additively generated by differences $(1-x^n)$, $n \in \mathbb{Z}$.

Note that the rings $\mathbb{Z}[\omega]$ and $\mathbb{Z}[x, x^{-1}]$ are not isomorphic, however they become isomorphic after the truncation. Namely for every $i \geq 0$ there is ring isomorphism

$$\phi_i\colon \mathbb{Z}[x, x^{-1}]/I_m^{i+1} \xrightarrow{\simeq} \mathbb{Z}[\omega]/I_a^{i+1}$$

defined by $\phi_i\colon x \mapsto (1-\omega)^{-1} = 1 + \omega + \ldots + \omega^i$ with the inverse defined by $\phi_i^{-1}\colon \omega \mapsto 1 - x^{-1}$. It is useful to keep the following picture in mind

$$\begin{tikzcd}
\mathbb{Z}[x, x^{-1}] \arrow[d, two heads] \arrow[dr] & \mathbb{Z}[\omega] \arrow[l] \arrow[d, two heads] \\
\mathbb{Z}[x, x^{-1}]/I_m^{i+1} \arrow[r, "\phi_i", "\simeq"'] & \mathbb{Z}[\omega]/I_a^{i+1}
\end{tikzcd}$$

observing that the inverse ϕ_i^{-1} can be lifted to the map $\mathbb{Z}[\omega] \to \mathbb{Z}[x, x^{-1}]$ but ϕ_i can't.

The example can be generalized as follows:

1.2. **Lemma.** [GaZ, 2.1] *Assume that Λ is a free abelian group of finite rank n. The rings $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$ become isomorphic after truncation. Namely, if $\{\omega_1, \ldots, \omega_n\}$ is a $\mathbb{Z}$-basis of Λ, then for every $i \geq 0$ there is a ring isomorphism*

$$\phi_i \colon \mathbb{Z}[\Lambda]/I_m^{i+1} \xrightarrow{\simeq} S^*(\Lambda)/I_a^{i+1}$$

defined by $\phi_i(1) = 1$ and

$$\phi_i(e^{\sum_{j=1}^n a_j\omega_j}) = \prod_{j=1}^n (1-\omega_j)^{-a_j}$$

with the inverse defined by $\phi_i^{-1}(\omega_j) = 1 - e^{-\omega_j}$.

Note that the map ϕ_i preserves the I-adic filtrations. Indeed, by definition $\phi_i(I_m^j) \subseteq I_a^j$ for every $0 \leq j \leq i$. Moreover, we have the following

1.3. **Lemma.** *The isomorphism ϕ_i restricted to the subsequent quotients I_m^i/I_m^{i+1} doesn't depend on the choice of a basis of Λ. Hence, there is an induced canonical isomorphism of graded rings*

$$\phi_* = \oplus_{i\geq 0}\phi_i : gr_m^*(\Lambda) \xrightarrow{\simeq} gr_a^*(\Lambda) = S^*(\Lambda).$$

Proof. Indeed, in this case we can define the inverse $\phi_i^{-1} \colon I_a^i/I_a^{i+1} \to I_m^i/I_m^{i+1}$ by

$$\phi_i^{-1}(\lambda_1\lambda_2\ldots\lambda_i) = (1-e^{-\lambda_1})(1-e^{-\lambda_2})\ldots(1-e^{-\lambda_i}).$$

It is well-defined since $(1-e^{-\lambda-\lambda'}) = (1-e^{-\lambda}) + (1-e^{-\lambda'})$ modulo I_m^2. $\square$

Consider the composite of the map ϕ_i with the projections

$$\phi^{(i)} \colon \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda]/I_m^{i+1} \xrightarrow{\phi_i} S^*(\Lambda)/I_a^{i+1} \to S^i(\Lambda).$$

The map $\phi^{(i)}$, and therefore ϕ_i, can be computed on generators e^λ, $\lambda \in \Lambda$ as follows:

Let $f(z) = \prod_j (1-\omega_j z)^{-a_j}$, where $\lambda = \sum_j a_j\omega_j$. Then

$$\phi^{(i)}(e^{\sum_j a_j\omega_j}) = \frac{1}{i!}\frac{d^i f(z)}{dz^i}\Big|_{z=0}$$

To compute the derivatives of $f(z)$ we observe that $f'(z) = f(z)g(z)$, where $g(z) = \sum_j a_j\omega_j(1-\omega_j z)^{-1}$ and $\frac{d^i\, g(z)}{d\, z^i} = \sum_j \frac{i!\, a_j\omega_j^{i+1}}{(1-\omega_j z)^{i+1}}$. Hence, starting with $g_0 = 1$ we obtain the following recursive formulas

$$\frac{d^i\, f(z)}{d\, z^i} = f(z)\cdot g_i(z), \text{ where } g_i(z) = g(z)g_{i-1}(z) + g_{i-1}'(z).$$

1.4. **Example.** For small values of i we obtain

i	$i!\cdot\phi^{(i)}(e^\lambda) =$
1	λ
2	$\lambda^2 + \lambda(2)$
3	$\lambda^3 + 3\lambda(2)\lambda + 2\lambda(3)$
4	$\lambda^4 + 6\lambda(4) + 6\lambda(2)\lambda^2 + 8\lambda(3)\lambda + 3\lambda(2)^2$

where given a presentation $\lambda = \sum_{j=1}^n a_{j,\lambda}\omega_j$, $a_{j,\lambda} \in \mathbb{Z}$ in terms of the basis $\{\omega_1, \omega_2, \ldots.\omega_n\}$, the character $\lambda(m)$, $m \geq 1$ is defined by

$$\lambda(m) = \sum_{j=1}^n a_{j,\lambda}\omega_j^m.$$

2. Invariants and exponents

Let W be a finite group which acts on a free abelian group Λ of finite rank by $\mathbb{Z}$-linear automorphisms. Consider the induced action of W on $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$. Observe that it is compatible with the I-adic filtrations, i.e. $W(I_m^i) \subseteq I_m^i$ and $W(I_a^i) \subseteq I_a^i$ for every $i \geq 0$.

Note that the isomorphisms ϕ_i and ϕ_i^{-1} are not necessarily W-equivariant. However, by Lemma 1.3 their restrictions to the subsequent quotients I_m^i/I_m^{i+1} and $I_a^i/I_a^{i+1} = S^i(\Lambda)$ are W-equivariant and we have

$$(I_m^i/I_m^{i+1})^W \simeq (I_a^i/I_a^{i+1})^W.$$

Let I_m^W denote the ideal of $\mathbb{Z}[\Lambda]$ generated by W-invariant elements from the augmentation ideal I_m, i.e., by elements from $\mathbb{Z}[\Lambda]^W \cap I_m$. Similarly, let I_a^W denote the ideal of $S^*(\Lambda)$ generated by W-invariant elements from I_a, i.e., by elements from $S^*(\Lambda)^W \cap I_a$.

For each $\chi \in \Lambda$ let $\rho(\chi) = \sum_{\lambda \in W(\chi)} e^\lambda$ denote the sum over all elements of the W-orbit of χ. Every element in I_m^W can be written as a finite linear combination with integer coefficients of the elements $\hat{\rho}(\chi) = \rho(\chi) - \epsilon_m(\rho(\chi))$, $\chi \in \Lambda$. Therefore, the ideal I_m^W is generated by the elements $\hat{\rho}(\chi)$, i.e.,

$$I_m^W = \langle \hat{\rho}(\chi) \mid \chi \in \Lambda \rangle.$$

The image of I_m^W by means of the composite

$$\mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda]/I_m^{i+1} \xrightarrow{\phi_i} S^*(\Lambda)/I_a^{i+1}.$$

is an ideal in $S^*(\Lambda)/I_a^{i+1}$ generated by the elements $\phi_i(\hat{\rho}(\chi))$, $\chi \in \Lambda$. Therefore, the image of I_m^W in $S^i(\Lambda)$ is the ith homogeneous component of the ideal generated by $\phi^{(j)}(\hat{\rho}(\chi))$, where $1 \leq j \leq i$, $\chi \in \Lambda$, i.e.

$$\phi^{(i)}(I_m^W) = \langle f \cdot \phi^{(j)}(\hat{\rho}(\chi)) \mid 1 \leq j \leq i,\ f \in S^{i-j}(\Lambda),\ \chi \in \Lambda \rangle_{\mathbb{Z}}.$$

We are ready now to introduce the central notion of the present paper:

2.1. **Definition.** We say that an action of W on Λ *has finite exponent in degree i* if there exists a non-zero integer N_i such that

$$N_i \cdot (I_a^W)^{(i)} \subseteq \phi^{(i)}(I_m^W),$$

where $(I_a^W)^{(i)} = I_a^W \cap S^i(\Lambda)$. In this case the g.c.d. of all such N_is will be called the *i-th exponent* of the W-action and will be denoted by τ_i.

Observe that if $\phi^{(i)}(I_m^W)$ is a subgroup of finite index in $(I_a^W)^{(i)}$, then τ_i is simply the exponent of $\phi^{(i)}(I_m^W)$ in $(I_a^W)^{(i)}$. Note also that by the very definition $\tau_0 = 1$ and $\tau_i \mid \tau_{i+1}$ for every $i \geq 0$.

3. Essential actions

In the present section we study W-actions that have no W-invariant linear forms, i.e. we assume that $\Lambda^W = 0$. In the theory of reflection groups such actions are called *essential* (see [B, V, §3.7] or [H]). Note that this immediately implies that $\tau_1 = 1$.

3.1. **Lemma.** *For every $\chi \in \Lambda$ and $m \in \mathbb{N}_+$ we have $\sum_{\lambda \in W(\chi)} \lambda(m) = 0$.*

Proof. Let $\omega_1, \omega_2, \dots .\omega_n$ be a $\mathbb{Z}$-basis of Λ. For $m \in \mathbb{N}_+$ we have

$$\sum_{\lambda \in W(\chi)} \lambda(m) = \sum_{\lambda \in W(\chi)} \big(\sum_{j=1}^{n} a_{j,\lambda}\omega_j^m\big) = \sum_{j=1}^{n} \big(\sum_{\lambda \in W(\chi)} a_{j,\lambda}\big)\omega_j^m.$$

In particular, for $m = 1$ we obtain

$$\sum_{\lambda \in W(\chi)} \lambda = \sum_{j=1}^{n} \big(\sum_{\lambda \in W(\chi)} a_{j,\lambda}\big)\omega_i.$$

Since $\Lambda^W = 0$, we have $\sum_{\lambda \in W(\chi)} \lambda = 0$. Since ω_j, $1 \leq j \leq n$ are $\mathbb{Z}$-free, we have $\sum_{\lambda \in W(\chi)} a_{j,\lambda} = 0$ for all $1 \leq j \leq n$. □

3.2. **Corollary.** *For every $\chi \in \Lambda$ we have*

$$\phi^{(2)}(\rho(\chi)) = \tfrac{1}{2} \sum_{\lambda \in W(\chi)} \lambda^2.$$

In particular, the quadratic form $\phi^{(2)}(\rho(\chi))$ is W-invariant, i.e.

$$\phi^{(2)}(\rho(\chi)) \in S^2(\Lambda)^W.$$

Proof. By the formula for $\phi^{(2)}$ in Example 1.4 and by Lemma 3.1 we obtain that

$$\phi^{(2)}\big(\sum_{\lambda \in W(\chi)} e^{\lambda}\big) = \tfrac{1}{2} \sum_{\lambda \in W(\chi)} (\lambda^2 + \lambda(2)) = \tfrac{1}{2} \sum_{\lambda \in W(\chi)} \lambda^2. \quad \square$$

3.3. **Corollary.** *If $S^2(\Lambda)^W = \langle q \rangle$ for some q, then $\phi^{(2)}(I_m^W)$ is a subgroup of finite index in $(I_a^W)^{(2)}$.*

Proof. The image of the ideal I_m^W is generated by $\phi^{(1)}(\rho(\chi))$ and $\phi^{(2)}(\rho(\chi))$. Since $\Lambda^W = 0$, $\phi^{(1)}(\rho(\chi)) = \sum_{\lambda \in W(\chi)} \lambda = 0$ and by Corollary 3.2, $\phi^{(2)}(I_m^W)$ is generated only by the W-invariant quadratic forms $\phi^{(2)}(\rho(\chi))$. For every $\chi \in \Lambda$ let

$$\phi^{(2)}(\rho(\chi)) = N_\chi \cdot q,\ N_\chi \in \mathbb{N}. \tag{1}$$

Then the subgroup $\phi^{(2)}(I_m^W)$ is a subgroup of $(I_a^W)^{(2)}$ of exponent

$$\tau_2 = \gcd_{\chi \in \Lambda} N_\chi. \quad \square$$

We now investigate the invariants of degree 3 and 4.

3.4. **Lemma.** *For every $\chi \in \Lambda$ we have*

$$\phi^{(3)}(\rho(\chi)) = \tfrac{1}{6} \sum_{\lambda \in W(\chi)} (\lambda^3 + 3\lambda(2)\lambda).$$

Proof. By the formula for $\phi^{(3)}$ in Example 1.4 and by Lemma 3.1 we obtain that

$$\phi^{(3)}(\rho(\chi)) = \tfrac{1}{6} \sum_{\lambda \in W(\chi)} (\lambda^3 + 3\lambda(2)\lambda + 2\lambda(3)) = \tfrac{1}{6} \sum_{\lambda \in W(\chi)} (\lambda^3 + 3\lambda(2)\lambda). \quad \square$$

3.5. **Lemma.** *For every $\chi \in \Lambda$ we have*

$$\phi^{(4)}(\rho(\chi)) = \tfrac{1}{24} \sum_{\lambda \in W(\chi)} [\lambda^4 + 6\lambda(2)\lambda^2 + 8\lambda(3)\lambda + 3\lambda(2)^2].$$

Proof. It follows from Example 1.4 and Lemma 3.1. □

4. The Dynkin index

In the present section we show that the action of the Weyl group W of a crystallographic root system Φ on the weight lattice Λ has finite exponent in degree 2 which coincides with the Dynkin index of the respective Lie algebra.

Let W be the Weyl group of a crystallographic root system Φ and let Λ be its weight lattice as defined in [H, §2.9]. Let $\{\omega_1, \ldots, \omega_n\}$ be a basis of Λ consisting of fundamental weights (here n is the rank of Φ).

The Weyl group W acts on $\lambda \in \Lambda$ by means of simple reflections

$$s_j(\lambda) = \lambda - \langle \alpha_j^\vee, \lambda \rangle \cdot \alpha_j, \quad j = 1 \ldots n$$

where $\alpha_j^\vee$ is the j-th simple coroot and $\langle -, - \rangle$ is the usual pairing. Note that $\langle \alpha_j^\vee, \omega_i \rangle = \delta_{ij}$, where δ_{ij} is the Kronecker symbol.

The subring of invariants $\mathbb{Z}[\Lambda]^W$ is the representation ring of the respective Lie algebra $\mathfrak{g}$. By a theorem of Chevalley it is the polynomial ring in fundamental representations $\rho(\omega_j) \in \mathbb{Z}[\Lambda]^W$, i.e.

$$\mathbb{Z}[\Lambda]^W \simeq \mathbb{Z}[\rho(\omega_1), \ldots, \rho(\omega_n)].$$

Observe that the dimension of the fundamental representation $\rho(\omega_j)$ equals to the number of elements in the orbit that is $\epsilon_m(\rho(\omega_j))$.

Therefore, the ideal I_m^W is generated by the elements $\hat{\rho}(\omega_j)$, $j = 1 \ldots n$ and its image $\phi^{(i)}(I_m^W)$ is the i-th homogeneous component of the ideal generated by $\phi^{(j)}(\rho(\omega_l))$, $1 \leq j \leq i$, $l = 1 \ldots n$.

4.1. **Lemma.** *We have $\Lambda^W = 0$ and hence also*

$$\phi^{(1)}(\mathbb{Z}[\Lambda]^W) = \phi^{(1)}(I_m^W) = 0.$$

Proof. Let $\eta \in \Lambda^W$. Since $\eta = s_{\alpha_j}(\eta) = \eta - \langle \eta, \alpha_j^\vee \rangle \alpha_j$ we have $\langle \eta, \alpha_j^\vee \rangle = \frac{2(\alpha_j, \eta)}{(\alpha_j, \alpha_j)} = 0$ for all simple roots α_j which implies that $\eta = 0$. $\square$

4.2. **Lemma.** *We have $S^2(\Lambda)^W = \langle q \rangle$.*

Proof. By [GN, Prop. 4] there exists an integer valued W-invariant quadratic form on Λ which has value 1 on short coroots. As the group $S^2(\Lambda)^W$ is identical to the group of all integral W-invariant quadratic forms on $T_* \otimes \mathbb{R}$, the result follows. $\square$

4.3. **Corollary.** *The image $\phi^{(2)}(I_m^W)$ is a subgroup of $(I_a^W)^{(2)}$ of finite index.*

Proof. This follows from Corollary 3.3 and Lemma 4.1. $\square$

We recall briefly the notion of indices of representations introduced by Dynkin [D, §2] (See also [BR]).

Let $f : \mathfrak{g} \to \mathfrak{g}'$ be a morphism between Lie algebras. Then there exists a unique number $j_f \in \mathbb{C}$, called the *Dynkin index of f*, satisfying

$$(f(x), f(y)) = j_f(x, y),$$

for all $x, y \in \mathfrak{g}$, where (–,–) is the Killing form on $\mathfrak{g}$ and $\mathfrak{g}'$ normalized such that $(\alpha, \alpha) = 2$ for any long root α. In particular, if $f : \mathfrak{g} \to \mathfrak{sl}(V)$ is a linear representation, j_f is a positive integer, called the *Dynkin index of the linear representation f*, defined by

$$\mathrm{tr}(f(x), f(y)) = j_f(x, y).$$

The *Dynkin index of* $\mathfrak{g}$ is defined to be the greatest common divisor of all the Dynkin indices of all linear representations of $\mathfrak{g}$. By [D, (2.24) and (2.25)], the Dynkin index of $\mathfrak{g}$ is the greatest common divisor of the Dynkin index of its fundamental representations. Moreover, all the Dynkin indices of the fundamental representations were calculated in [D, Table 5].

Using the $\mathfrak{sl}_2$-representation theory, the Dynkin index of a linear representation $f:\mathfrak{g}\to\mathfrak{sl}(V)$ can be described as follows. Let α be a long root. For the formal character $\mathrm{ch}(V)=\sum_\lambda n_\lambda e^\lambda$, one has (see [LS, Lemma 2.4] or [KNR, 5.1 and Lemma 5.2])

$$j_f=\frac{1}{2}\sum_\lambda \langle\lambda,\alpha^\vee\rangle^2. \tag{2}$$

4.4. **Theorem.** *The integers $N(\omega_j)$ for the j-th fundamental weight as defined in (1) coincide with the Dynkin index of the fundamental representation with highest weight ω_j. In particular, the second exponent τ_2 coincides with the Dynkin index of $\mathfrak{g}$.*

Proof. To find the precise value of τ_2 we use the explicit formula for $\phi^{(2)}$, that is

$$\phi^{(2)}(\rho(\chi))=\tfrac{1}{2}\sum_{\lambda\in W(\chi)}\lambda^2.$$

We know that τ_2 is the greatest common divisor of the integers $N_j=N_{\omega_j}$ using the notation of the proof of Corollary 3.3, where ω_j is the j-th fundamental weight of $\mathfrak{g}$. As the Dynkin index is the greatest common divisor of the Dynkin indices of the fundamental representations ω_j, it suffices to show that N_j coincides with the Dynkin index of the representation V_j corresponding to ω_j. We can view $\phi^{(2)}(\rho(\chi))$ for $\chi=\omega_j$ as a function on the lattice $\mathfrak{h}_\mathbb{Z}=\mathrm{Span}_\mathbb{Z}\{\alpha^\vee\mid\alpha\in\Phi\text{ long}\}$. Since V_j has character $\mathrm{ch}(V_j)=\sum_{\lambda\in W(\omega_j)}e^\lambda$, by (2) the Dynkin index of the representation V_j is $\frac{1}{2}\sum_{\lambda\in W(\omega_j)}\langle\lambda,\alpha^\vee\rangle^2$, where α is any long root in Φ. Thus, $\phi^{(2)}(\rho(\omega_j))$ is the constant function with value N_j. $\square$

We note that a different proof of Theorem 4.4 was given in [GaZ, §2].

5. Exponents of degrees 3 and 4

In the present section we show that $\tau_2=\tau_3=\tau_4$ for all crystallographic root systems

Let $S=\{\lambda_1,\ldots,\lambda_r\}$ be a finite set of weights. We denote by $-S$ the set of opposite weights $\{-\lambda_1,\ldots,-\lambda_r\}$, by S_+ the set of sums $\{\lambda_i+\lambda_j\}_{i<j}$, by S_- the set of differences $\{\lambda_i-\lambda_j\}_{i<j}$ and by $S_\pm$ the disjoint union $S_+\amalg S_-$. By definition we have $|S_+|=|S_-|=\binom{r}{2}$.

Using the fact that $(\lambda+\lambda')(m)=\lambda(m)+\lambda'(m)$ for every $\lambda,\lambda'\in\Lambda$ and $m\geq 0$ we obtain the following lemma which will be extensively used in the computations

5.1. **Lemma.** *(i) For every integer $m_1,m_2,x,y\geq 0$ and a finite subset $S\subset\Lambda$ we have*

$$\sum_{\lambda\in S\amalg -S}\lambda(m_1)^x\lambda(m_2)^y=(1+(-1)^{x+y})\sum_{\lambda\in S}\lambda(m_1)^x\lambda(m_2)^y.$$

In particular, $\sum_{\lambda\in S\amalg -S}\lambda(2)\lambda^2=0$.

(ii) For every subset $S \subset \Lambda$ with $|S| = r$ and for every $m_1, m_2 \geq 0$ we have

$$\sum_{\lambda\in S_+} \lambda(m_1)\lambda(m_2) = (r-1)\sum_{\lambda\in S} \lambda(m_1)\lambda(m_2) + \sum_{i\neq j} \lambda_i(m_1)\lambda_j(m_2) \text{ and}$$

$$\sum_{\lambda\in S_-} \lambda(m_1)\lambda(m_2) = (r-1)\sum_{\lambda\in S} \lambda(m_1)\lambda(m_2) - \sum_{i\neq j} \lambda_i(m_1)\lambda_j(m_2).$$

In particular, this implies that $\sum_{\lambda\in S_\pm} \lambda(m_1)\lambda(m_2) = 2(r-1)\sum_{\lambda\in S} \lambda(m_1)\lambda(m_2)$.

A_n**-case.** Let Φ be of type A_n for $n \geq 3$. We denote the canonical basis of $\mathbb{R}^{n+1}$ by e_i with $1 \leq i \leq n+1$. According to [H, §3.5 and §3.12] the basic polynomial invariants of the W-action on Λ (algebraically independent homogeneous generators of $S^*(\Lambda)^W$ as a $\mathbb{Q}$-algebra) are given by the symmetric power sums

$$q_i := e_1^i + \dots + e_{n+1}^i, \quad 2 \leq i \leq n+1.$$

Let s_i denote the ith elementary symmetric function in $e_1, \dots, e_{n+1}$. Using the classical identities

$$q_1 = s_1, \quad q_i = s_1 q_{i-1} - s_2 q_{i-2} + \ldots + (-1)^i s_{i-1} q_1 + (-1)^{i+1} i \cdot s_i, \quad 1 < i < n+1$$

and the fact that $s_1 = 0$, we obtain that

$$q_2/2 = -s_2,\ q_3/3 = s_3, \text{ and } q_4/2 = s_2^2 - 2s_4.$$

generate (with integral coefficients) the ideal I_a^W up to degree 4.

The fundamental weights of Φ can be expressed as follows

$$\omega_1 = e_1,\ \omega_2 = e_1 + e_2,\ \dots, \omega_{n-1} = e_1 + \ldots + e_{n-1},\ \omega_n = -e_{n+1},$$

where $e_1 + e_2 + \ldots + e_{n+1} = 0$. The orbits of ω_1, $\omega_1 + \omega_n$, ω_n and ω_2, ω_{n-1} under the action of the Weyl group $W = S_{n+1}$ are given by

$$W(\omega_1) = \{e_1, \dots, e_{n+1}\} = -W(\omega_n),\ W(\omega_1 + \omega_n) = \{e_i - e_j\}_{i\neq j} \text{ and}$$
$$W(\omega_2) = \{e_i + e_j\}_{i<j} = -W(\omega_{n-1}).$$

Therefore, $W(\omega_1 + \omega_n) = S_- \amalg -S_-$ and $W(\omega_2) = S_+$, where $S = W(\omega_1)$.

Applying Lemma 3.5 and Lemma 5.1 we obtain that

$$\phi^{(4)}(\rho(\omega_1) + \rho(\omega_n)) = \tfrac{1}{12} \sum_{\lambda\in S} (\lambda^4 + 8\lambda(3)\lambda + 3\lambda(2)^2) \text{ and}$$

$$\phi^{(4)}(\rho(\omega_1 + \omega_n) + \rho(\omega_2) + \rho(\omega_{n-1})) = \tfrac{1}{24} \sum_{\lambda\in S_\pm \amalg -S_\pm} (\lambda^4 + 8\lambda(3)\lambda + 3\lambda(2)^2) =$$
$$= \tfrac{1}{24} \sum_{\lambda\in S_\pm \amalg -S_\pm} \lambda^4 + \tfrac{n}{6} \sum_{\lambda\in S} (8\lambda(3)\lambda + 3\lambda(2)^2).$$

Then the difference

$$\phi^{(4)}(\rho(\omega_1 + \omega_n) + \rho(\omega_2) + \rho(\omega_{n-1})) - 2n \cdot \phi^{(4)}(\rho(\omega_1) + \rho(\omega_n)) =$$
$$= \tfrac{1}{24} \sum_{\lambda\in S_\pm \amalg -S_\pm} \lambda^4 - \tfrac{n}{6} \sum_{\lambda\in S} \lambda^4 = \tag{3}$$

is a symmetric function in $e_1, \dots, e_{n+1}$ and, therefore, it can be always written as a polynomial in q_is. Indeed, since

$$\sum_{\lambda\in S_\pm \amalg -S_\pm} \lambda^4 = 2\sum_{i<j} ((e_i + e_j)^4 + (e_i - e_j)^4) = 4n \sum_{\lambda\in S} \lambda^4 + 24 \sum_{i<j} e_i^2 e_j^2,$$

the difference (3) equals
$$= \sum_{i<j} e_i^2 e_j^2 = (q_2^2 - q_4)/2.$$

5.2. **Lemma.** *For a root system of type A_n, $n \geq 2$, we have $\tau_2 = \tau_3 = \tau_4 = 1$.*

Proof. It is enough to show that the generators $q_2/2$, $q_3/3$ and $q_4/2$ are in the ideal generated by the image of $\phi^{(i)}$, $i \leq 4$.

By Corollary 3.2 we have $\phi^{(2)}(\rho(\omega_1)) = \frac{1}{2}\sum_{\lambda\in S}\lambda^2 = q_2/2$. By Lemma 3.4 we have $q_3/3 = \phi^{(3)}(\rho(\omega_1)) - \phi^{(3)}(\rho(\omega_n))$ (see also [GaZ, §1C]). If Φ is of type A_2, then $s_4 = 0$ and, hence, $q_4 = q_2^2/2$. If Φ is of type A_n, $n \geq 3$, then by (3) the generator $q_4/2$ belongs to the ideal generated by the images of $\phi^{(2)}$ and $\phi^{(4)}$. $\square$

5.3. **Lemma.** *For any crystallographic root system Φ the third exponent τ_3 of the W-action coincides with τ_2 (the Dynkin index).*

Proof. If Φ is of type A_n, this follows from Lemma 5.2; for the other types there are no basic polynomial invariants of degree 3 [H, §3.7 Table 1]. Therefore, $\tau_3 = \tau_2$. $\square$

B_n, C_n and D_n cases. Let Φ be of type B_n or C_n for $n \geq 2$ or of type D_n for $n \geq 4$. We denote the canonical basis of $\mathbb{R}^n$ by e_i with $1 \leq i \leq n$. By [H, §3.5 and §3.12] the basic polynomial invariants of the W-action on Λ are given by even power sums
$$q_{2i} := e_1^{2i} + \cdots + e_n^{2i}, \quad 1 \leq i \leq n.$$

The first two fundamental weights of Φ are given by $\omega_1 = e_1$, $\omega_2 = e_1 + e_2$ and their W-orbits are
$$W(\omega_1) = \{\pm e_1, \ldots, \pm e_n\} \text{ and } W(\omega_2) = \{\pm e_i \pm e_j\}_{i<j}.$$
Hence $W(\omega_1) = S \amalg -S$ and $W(\omega_2) = S_\pm \amalg -S_\pm$, where $S = \{e_1, \ldots, e_n\}$.

Applying Lemma 3.5 and Lemma 5.1 we obtain that
$$\phi^{(4)}(\rho(\omega_1)) = \tfrac{1}{12}\sum_{\lambda\in S}\lambda^4 + \tfrac{1}{12}\sum_{\lambda\in S}(8\lambda(3)\lambda + 3\lambda(2)^2) \text{ and}$$
$$\phi^{(4)}(\rho(\omega_2)) = \tfrac{1}{24}\sum_{\lambda\in S_\pm \amalg -S_\pm}\lambda^4 + \tfrac{n-1}{6}\sum_{\lambda\in S}(8\lambda(3)\lambda + 3\lambda(2)^2).$$
Then similar to the A_n-case we obtain
$$\phi^{(4)}(\rho(\omega_2)) - 2(n-1)\phi^{(4)}(\rho(\omega_1)) = (q_2^2 - q_4)/2, \tag{4}$$
where $q_i = e_1^i + \ldots + e_n^i$.

5.4. **Lemma.** *For a root system of type B_n or C_n, $n \geq 2$ or D_n, $n \geq 4$ we have $\tau_4 = \tau_2$.*

Proof. It is enough to show that $q_4/2$ is in the ideal generated by the image of $\phi^{(2)}$ and $\phi^{(4)}$.

By Corollary 3.2 we have $\phi^{(2)}(\rho(\omega_1)) = \sum_{\lambda\in S}\lambda^2 = q_2$. Therefore, by (4)
$$q_4/2 = (q_2/2)\cdot\phi^{(2)}(\rho(\omega_1)) - \phi^{(4)}(\rho(\omega_2)) + 2(n-1)\phi^{(4)}(\rho(\omega_1))$$
and the proof is finished. $\square$

5.5. **Theorem.** *For any crystallographic root system Φ we have $\tau_2 = \tau_3 = \tau_4$.*

Proof. The equality $\tau_2 = \tau_3$ is proven in Lemma 5.3. If Φ is of type A_n, $\tau_4 = 1$ follows from Lemma 5.2. If Φ is of type B_n, C_n or D_n, $\tau_4 = \tau_2$ follows from Lemma 5.4. For all other types $\tau_4 = \tau_2$ since there are no basic polynomial invariants of degree 3 and 4 (see [H, §3.7 Table 1]). □

6. Torsion in the Grothendieck γ-filtration

The goal of the present section is to provide geometric interpretation (see (6)) of the map ϕ_i and the exponents τ_i.

Let G be a simple simply-connected Chevalley group over a field k. We fix a maximal split torus T of G and a Borel subgroup $B \supset T$. Let Λ be the group of characters of T. Since G is simply-connected, Λ coincides with the weight lattice of G.

Let X denote the variety of Borel subgroups of G (conjugate to B). Consider the Chow ring $\mathrm{CH}^*(X)$ of algebraic cycles modulo rational equivalence and the Grothendieck ring $K_0(X)$. Following [De74, §1] to every character $\lambda \in \Lambda$ we may associate the line bundle $\mathcal{L}(\lambda)$ over X. It induces the ring homomorphisms (called the characteristic maps)

$$\mathfrak{c}_a\colon S^*(\Lambda) \to \mathrm{CH}^*(X) \text{ and } \mathfrak{c}_m\colon \mathbb{Z}[\Lambda] \twoheadrightarrow K_0(X)$$

by sending $\lambda \mapsto c_1(\mathcal{L}(\lambda))$ and $e^\lambda \mapsto [\mathcal{L}(\lambda)]$ respectively. Note that the map $\mathfrak{c}_a$ is an isomorphism in codimension one, hence, giving

$$\mathfrak{c}_a\colon S^1(\Lambda) = \Lambda \overset{\simeq}{\to} Pic(X) = CH^1(X)$$

and the map $\mathfrak{c}_m$ is surjective. Let W be the Weyl group and let I_a^W and I_m^W denote the respective W-invariant ideals. Then according to [De73, §4 Cor.2,§9] and [CPZ, §6]

$$\ker \mathfrak{c}_m = I_m^W \tag{5}$$

and $\ker \mathfrak{c}_a$ is generated by elements of $S^*(\Lambda)$ such that their multiples are in I_a^W.

Consider the Grothendieck γ-filtration on $K_0(X)$ (see [GaZ, §1]). Its ith term is an ideal generated by products

$$\gamma^i(X) := \langle (1-[\mathcal{L}_1^\vee])(1-[\mathcal{L}_2^\vee])\cdot\ldots\cdot(1-[\mathcal{L}_i^\vee])\rangle,$$

where $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_i$ are line bundles over X. Consider the ith subsequent quotient $\gamma^i(X)/\gamma^{i+1}(X)$. The usual Chern class c_i induces a group homomorphism $c_i\colon \gamma^i(X)/\gamma^{i+1}(X) \to CH^i(X)$.

6.1. **Proposition.** *For every $i \geq 0$ there is a commutative diagram of group homomorphisms*

$$\begin{tikzcd} I_m^i/I_m^{i+1} \arrow[r, "(-1)^{i-1}(i-1)!\cdot\phi_i"] \arrow[d, "\mathfrak{c}_m"] & S^i(\Lambda) \arrow[d, "\mathfrak{c}_a"] \\ \gamma^i(X)/\gamma^{i+1}(X) \arrow[r, "c_i"] & \mathrm{CH}^i(X) \end{tikzcd} \tag{6}$$

Proof. Indeed, the γ-filtration on $K_0(X)$ is the image of the I_m-adic filtration on $\mathbb{Z}[\Lambda]$, i.e. $\gamma^i(X) = \mathfrak{c}_m(I_m^i)$ for every $i \geq 0$. The Proposition then follows from the identity

$$c_i\Big((1-[\mathcal{L}_1^\vee])(1-[\mathcal{L}_2^\vee])\ldots(1-[\mathcal{L}_i^\vee])\Big) = (-1)^{i-1}(i-1)!\cdot c_1(\mathcal{L}_1)c_1(\mathcal{L}_2)\ldots c_1(\mathcal{L}_i),$$

where $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_i$ are line bundles over X and $\mathcal{L}_i^\vee$ denotes the dual of $\mathcal{L}_i$. $\square$

6.2. **Remark.** Note that $\mathbb{Z}[\Lambda]$ can be identitfied with the T-equivariant K_0 of a point $pt = Spec\ k$ and $S^*(\Lambda)$ with the T-equivariant CH of a point (see [GiZ]). The maps $\mathfrak{c}_a$ and $\mathfrak{c}_m$ then can be identified with the pull-backs $K_T(pt) \to K_T(G)$ and $CH_T(pt) \to CH_T(G)$ induced by the structure map $G \to pt$.

In view of these identifications the map ϕ_i can be viewed as an equivariant analogue of the Chern class map c_i.

Consider the diagram (6) with $\mathbb{Q}$-coefficients. In this case the Chern class map c_i will become an isomorphism (by the Riemann-Roch theorem), the characteristic map $\mathfrak{c}_a$ will turn into a surjection and the map $(-1)^{i-1}(i-1)! \cdot \phi_i$ will be an isomorphism as well. In view of (5) we obtain an isomorphism

$$\phi^{(i)} \otimes \mathbb{Q} \colon I_m^W \cap I_m^i / I_m^W \cap I_m^{i+1} \otimes \mathbb{Q} \longrightarrow (I_a^W)^{(i)} \otimes \mathbb{Q}$$

on the kernels of $\mathfrak{c}_m$ and $\mathfrak{c}_a$. By the very definition of the exponents τ_i this implies that

6.3. **Corollary.** *The action of the Weyl group of a crystallograhic root system has finite exponent τ_i for every i.*

We are now ready to prove the main result of this section

6.4. **Theorem.** *The integer $\tau_i \cdot (i-1)!$ annihilates the torsion of the ith subsequent quotient $\gamma^i(X)/\gamma^{i+1}(X)$ of the γ-filtration on $K_0(X)$ for $i = 3, 4$.*

6.5. **Remark.** Note that by [SGA, Exposé XIV, 4.5] for groups of types A_n and C_n the quotients $\gamma^i(X)/\gamma^{i+1}(X)$ have no torsion.

Proof. Assume that α is a torsion element in $\gamma^i(X)/\gamma^{i+1}(X)$. Then $c_i(\alpha) = 0$ since $CH^i(G/B)$ has no torsion. Let $\tilde\alpha$ be a preimage of α via $\mathfrak{c}_m$ in $I_m^i/I_m^{i+1} \subseteq \mathbb{Z}[\Lambda]/I_m^{i+1}$. By the same analysis as in [GaZ, §1B, §1C] one can show that $\ker(\mathfrak{c}_a)^{(i)} = (I_a^W)^{(i)}$ for $i \le 4$. By (6) we obtain that

$$(i-1)!\, \phi_i(\tilde\alpha) \in (I_a^W)^{(i)}$$

By definition of the index τ_i we have

$$\tau_i \cdot (i-1)!\, \phi_i(\tilde\alpha) = \phi_i(\beta), \text{ where } \beta \in I_m^W/I_m^{i+1} \cap I_m^W.$$

Applying ϕ_i^{-1} to the both sides we obtain

$$\tau_i \cdot (i-1)! \cdot \tilde\alpha = \beta \in I_m^W/I_m^{i+1} \cap I_m^W$$

Applying $\mathfrak{c}_m$ to the both sides and observing that $I_m^W = \ker \mathfrak{c}_m$ we obtain that $\tau_i \cdot (i-1)! \cdot \alpha = 0$. $\square$

Let ${}_\xi X$ be a twisted form of the variety X by means of a cocycle $\xi \in Z^1(k, G)$. By [P, Thm. 2.2.(2)] the restriction map $K_0({}_\xi X) \to K_0(X)$ (here we identify $K_0(X)$ with the $K_0(X \times_k \bar k)$ over the algebraic closure $\bar k$) is an isomorphism. Since the characteristic classes commute with restrictions, this induces an isomorphism between the γ-filtrations, i.e. $\gamma^i({}_\xi X) \simeq \gamma^i(X)$ for every $i \ge 0$, and between the respective quotients

$$\gamma^i({}_\xi X)/\gamma^{i+1}({}_\xi X) \simeq \gamma^i(X)/\gamma^{i+1}(X) \quad \text{for every } i \ge 0.$$

In view of this fact Theorem 6.4 imply that

6.6. **Corollary.** *Let G be a split simple simply connected group of type B_n ($n \geq 3$) or D_n ($n \geq 4$). Then for every $\xi \in Z^1(k,G)$ the torsion in $\gamma^4({}_\xi X)/\gamma^5({}_\xi X)$ is annihilated by* 12.

Consider the topological filtration on $K_0(Y)$ given by the ideals

$$\tau^i(Y) := \langle [\mathcal{O}_V] \mid V \hookrightarrow Y,\, codim_V Y \geq i\rangle.$$

It is known (see [GaZ, §2]) that $\gamma^i(Y) \subseteq \tau^i(Y)$ for every $i \geq 0$.

6.7. **Corollary.** *In the notation of Corollary 6.6 assume in addition that the induced map*

$$\gamma^4({}_\xi X)/\gamma^5({}_\xi X) \to \tau^4({}_\xi X)/\tau^5({}_\xi X)$$

is surjective. Then the 2-torsion of $\mathrm{CH}^4({}_\xi X)$ is annihilated by 8.

Proof. By the Riemann-Roch theorem [F, Ex.15.3.6], the composition

$$\mathrm{CH}^4({}_\xi X) \twoheadrightarrow \tau^4({}_\xi X)/\tau^5({}_\xi X) \xrightarrow{c_4} \mathrm{CH}^4({}_\xi X)$$

is the multiplication by $(-1)^{4-1}(4-1)! = -6$, where the first map is surjective. Hence, the torsion subgroup of $\mathrm{CH}^4({}_\xi X)$ is annihilated by 72 and so the result follows. □

7. 'The Dynkin index' in the H_2 case

Note that the notion of an exponent τ_i can be defined over a unique factorisation domain in the same way. As an example we compute the second exponent τ_2 for the action of the Weyl group of of the non-crystallographic root system H_2 over the base ring $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$, hence, giving rise to an interesting question about its geometric/Lie algebra interpretation.

7.1. **Theorem.** *For the non-crystallographic root system $H_2 := I_2(5)$, the second exponent τ_2 is $\sqrt{5}$.*

Proof. We follow the notations in [CMP]. In the root system H_2, the Weyl group W is the dihedral group of order 10 and M is the $\mathbb{Z}[\tau]$-lattice generated by two simple roots α_1 and α_2, where $\tau = (1+\sqrt{5})/2$. Observe that $\mathbb{Z}[\tau]$ is an Euclidean domain.

The dual basis $\{\omega_1, \omega_2\}$ is defined by

$$\begin{cases} \omega_1 = \frac{1}{3-\tau}(2\alpha_1 + \tau\alpha_2) \\ \omega_2 = \frac{1}{3-\tau}(\tau\alpha_1 + 2\alpha_2) \end{cases} \text{or} \begin{cases} \alpha_1 = 2\omega_1 - \tau\omega_2 \\ \alpha_2 = -\tau\omega_1 + 2\omega_2 \end{cases}$$

One computes the orbits of ω_1 and ω_2 as follows:

$$W(\omega_1) = \{\omega_1, -\omega_2, -\omega_1 + \tau\omega_2, -\tau\omega_1 + \omega_2, \tau\omega_1 - \tau\omega_2\},$$
$$W(\omega_2) = -W(\omega_1).$$

As the action of W on M is essential, by Corollary 3.2, we have

$$\begin{aligned}\phi^{(2)}(\rho(\omega_2)) = \phi^{(2)}(\rho(\omega_1)) &= \frac{1}{2}(\omega_1^2 + \omega_2^2 + (\omega_1 - \tau\omega_2)^2 + (\tau\omega_1 - \omega_2)^2 + (\tau\omega_1 - \tau\omega_2)^2) \\ &= (1+\tau^2)\omega_1^2 + (1+\tau^2)\omega_2^2 - (2\tau + \tau^2)\omega_1\omega_2. \qquad (7)\end{aligned}$$

Since $\phi^{(2)}(\rho(\omega_2))$ is W-invariant by Corollary 3.2, we have

$$\tau_2 = \gcd(1+\tau^2, 2\tau+\tau^2) = \gcd(2+\tau, 2\tau - 1).$$

But $2\tau - 1 = \sqrt{5}$ is a prime in $\mathbb{Z}[\tau]$, and we have $2 + \tau = (2\tau - 1)\tau$ proving that $\tau_2 = \sqrt{5}$. □

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